REPRESENTATION THEORY AND WIGNER–RACAH ALGEBRA OF THE SU(2) GROUP IN A NONCANONICAL BASIS

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It is a real pleasure to dedicate this paper to Professor Josef Paldus on the occasion of his 70th anniversary in recognition of his important contribution to various domains of the quantum theory of molecular electronic structure. The quantum chemistry community greatly benefits from his works on nonrelativistic and relativistic electronic systems via second quantization and field-theoretical approaches, diagrammatic methods, Lie-like and Clifford-like algebraic techniques. His numerous results on the many-electron correlation problem, on the configuration interaction method, on the coupled-cluster theory, and on density matrix calculations should be a source of inspiration for young theoretical chemists. The present author is very indebted to Prof. Paldus for interesting discussions and the kind hospitality extended to him on the occasion of several fruitful visits at the Mathematics Department of the University of Waterloo. Merci beaucoup Jo.

The Lie algebra su(2) of the classical group SU(2) is built from two commuting quon algebras for which the deformation parameter is a common root of unity. This construction leads to (i) a not very well-known polar decomposition of the generators J_- and J_+ of the SU(2) group, with $J_+ = J_-^\dagger = HU_r$ where H is Hermitean and U_r unitary, and (ii) an alternative to the $\{J^2, J_z\}$ quantization scheme, viz., the $\{J^2, U_r\}$ quantization scheme. The representation theory of the SU(2) group can be developed in this nonstandard scheme. The key ideas for developing the Wigner–Racah algebra of the SU(2) group in the $\{J^2, U_r\}$ scheme are given. In particular, some properties of the coupling and recoupling coefficients as well as the Wigner–Eckart theorem in the $\{J^2, U_r\}$ scheme are examined in great detail.
Keywords: Wigner–Eckart theorem; Hermitean; Oscillator algebras; Quantum mechanics; Quantum chemistry.

The concepts of symmetry (introduced in a group theoretical context in the 1930's), of supersymmetry (introduced in a supergroup context in the 1970's) and of deformations (introduced in a bi-algebra context in the 1980's) are of paramount importance for quantum chemistry and/or quantum physics. These concepts are often used in the exploration of dynamic systems as for example the Coulomb system and the oscillator system which can be viewed as two paradigms for the study of atomic and molecu-

lar interactions[1,2]. In these directions, the works of Paldus[3] and its collaborators on the second quantization method, the unitary group approach and its extension by means of Clifford algebras proved to be very useful in numerous domains of theoretical chemistry.

In recent years, the use of deformed oscillator algebras proved to be useful for many applications of quantum mechanics. For instance, one- and two-parameter deformations of oscillator algebras and Lie algebras were applied to statistical mechanics[4] and to molecular and nuclear physics[5].

It is the purpose of this work to apply deformed oscillator algebras or quon algebras to the representation theory and the Wigner–Racah algebra of the SU(2) group. The notion of deformation is very familiar to the theoretician. In this connection, quantum mechanics may be considered as a deformation (the deformation parameter being the rationalized Planck constant $\hbar$) of classical mechanics. In the same vein, relativistic mechanics is, to some extent, another deformation (with the inverse of the velocity of light c^{-1} as deformation parameter) of classical mechanics. The idea of a deformation of oscillator algebra and of Lie algebra also relies on the introduction of a deformation parameter q such that the limiting situation where $q = 1$ corresponds to the nondeformed algebraic structure.

The organization of this paper is as follows. Section 1 is devoted to some generalities on the notion of a Wigner–Racah algebra of a finite or compact group. In Section 2, we construct the Lie algebra of SU(2) from two quon algebras A_1 and A_2 corresponding to the same deformation parameter q taken as a root of unity. Section 3 deals with an alternative to the $\{J^2, J_z\}$ scheme of SU(2), viz. the $\{J^2, U_r\}$ scheme, and with the basic elements for the representation theory of SU(2) in this scheme. Finally, we develop in Section 4 the Wigner–Racah algebra of SU(2) in the $\{J^2, U_r\}$ scheme.

Throughout the present work, we use the notation $[A, B]$ for the commutator of A and B. As usual, z^* denotes the complex conjugate of the number z and $A^\dagger$ stands for the Hermitean conjugate of the operator A.

1. WIGNER–RACAH ALGEBRA OF SU(2)

The mathematical structure of a Wigner–Racah algebra (WRa) associated with a group takes its origin in the works by Wigner[6] on a simply reducible group, with emphasis on the ordinary rotation group, and by Racah[7] on chains of groups of type SU($2l + 1$) ⊃ SO($2l + 1$) ⊃ SO(3), mainly with $l = 2, 3$. From a practical point of view, the WRa of a group deals with the algebraic relations satisfied by its coupling and recoupling coefficients. From a more theoretical point of view, the WRa of a finite or compact group can be

defined to be the infinite-dimensional Lie algebra spanned by the Wigner unit operators (i.e., the operators whose matrix elements are the coupling or Clebsch–Gordan or Wigner coefficients of the group)[8].

The WRa of the SU(2) group is well known. It is generally developed in the standard basis $\{|jm\rangle : 2j \in \mathbf{N}, m = -j, -j + 1, ..., j\}$ arising in the simultaneous diagonalization of the Casimir operator J^2 and of one generator, say J_z, of SU(2). Besides this basis, there exist several other bases. Indeed, any change of basis of type

$$|j\mu\rangle = \sum_{m=-j}^{j} |jm\rangle \langle jm|j\mu\rangle \tag{1}$$

(where the $(2j + 1) \times (2j + 1)$ matrix with elements $\langle jm|j\mu\rangle$ is an arbitrary unitary matrix) defines another acceptable basis for the WRa of SU(2). In this basis, the matrices of the irreducible representation classes of SU(2) take a new form as well as the coupling coefficients (and the associated 3-jm symbols). As a matter of fact, the coupling coefficients $(j_1 j_2 m_1 m_2|jm)$ are simply replaced by

$$(j_1 j_2 \mu_1 \mu_2 | j\mu) = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m=-j}^{j} (j_1 j_2 m_1 m_2 | jm)$$

$$\langle j_1 m_1 | j_1 \mu_1 \rangle^* \langle j_2 m_2 | j_2 \mu_2 \rangle^* \langle jm | j\mu \rangle \tag{2}$$

when passing from the $\{jm\}$ quantization to the $\{j\mu\}$ quantization while the recoupling coefficients, and the corresponding $3(n - 1) - j$ symbols, for the coupling of n ($n > 2$) angular momenta remain invariant.

The various bases for SU(2) may be classified into two types: group–subgroup type and nongroup–subgroup type. The standard basis corresponds to a group–subgroup type basis associated with the chain of groups SU(2) ⊃ U(1). Another group–subgroup type basis may be obtained by replacing U(1) by a finite group G^* (generally the double, i.e., spinor group, of a point group G of molecular or crystallographic interest). Among the SU(2) ⊃ G^* bases, we may distinguish: the weakly symmetry-adapted bases for which the basis vectors are eigenvectors of J^2 and of the projection operators of G^* (e.g., see[9]) and the strongly symmetry-adapted bases for which the basis vectors are eigenvectors of J^2 and of an operator defined in the enveloping algebra of SU(2) and invariant under the group G (e.g., see[10]). We shall see that the basis for SU(2) described in the present paper interpolates between the group–subgroup type and the nongroup–subgroup type.

2. A QUON REALIZATION OF THE ALGEBRA su(2)

Two Quon Algebras

The concept of quon takes its origin in the replacement of the commutation (sign –) and anticommutation (sign +) relations

$$a_-a_+ \pm a_+a_- = 1 \tag{3}$$

by the relation

$$a_-a_+ - qa_+a_- = 1 \tag{4}$$

where q is a constant. Following the works[11], we define two commuting quon algebras $A_i = \{a_{i-}, a_{i+}, N_i\}$ with $i = 1$ and 2 by

$$a_{i-}a_{i+} - qa_{i+}a_{i-} = 1 \; , \qquad [N_i, a_{i\pm}] = \pm a_{i\pm} \; , \qquad N_i^\dagger = N_i \tag{5}$$

$$(a_{i+})^k = (a_{i-})^k = 0 \tag{6}$$

$$\forall x_1 \in A_1 \; , \quad \forall x_2 \in A_2 : [x_1, x_2] = 0 \tag{7}$$

where

$$q = \exp\left(\frac{2\pi i}{k}\right) \quad \text{with} \quad k \in \mathbf{N} \setminus \{0,1\} \; . \tag{8}$$

Equation (5) corresponds to the a la Arik and Coon[11] relations defining a quon algebra except that, in the present work, q is a root of unity instead of being a positive real number. The deformation parameter q is the same for each of the algebras A_1 and A_2 so that A_1 and A_2 can be considered as two copies of the same quon algebra. Equation (6) constitutes nilpotency conditions which are indeed compatibility relations to account for the fact that q is not a positive number (remember $q^k = 1$). Equation (7) reflects the commutativity of the algebras A_1 and A_2. The generators $a_{i\pm}$ and N_i of A_1 and A_2 are linear operators. As in the classical case $q = 1$, we say that a_{i+} is a creation operator, a_{i-} an annihilation operator and N_i a number operator (with $i = 1, 2$). However, the operator a_{i+} cannot be considered as the adjoint of the operator a_{i-} except for $k = 2$ and $k \to \infty$. In contrast, the operator N_i can be taken to be a Hermitean operator for any value of k in $\mathbf{N} \setminus \{0,1\}$. It should

be observed that N_i is different from $a_{i+}a_{i-}$ except for $k = 2$ and $k \to \infty$. Note that the case $k = 2$ ($\Rightarrow q = -1$) corresponds to fermion operators and the case $k \to \infty$ ($\Rightarrow q = -1$) to boson operators. In other words, each of the algebras A_i describes fermions for $q = -1$ and bosons for $q = 1$ with $N_i = a_{i+}a_{i-}$ for fermions and bosons ($i = 1, 2$).

To close this subsection, let us mention that algebras similar to A_1 and A_2 with $N_1 \equiv N_2$ were introduced by Daoud, Hassouni and Kibler[11] for defining *k*-fermions which are, like anyons, objects interpolating between fermions (corresponding to $k = 2$) and bosons (corresponding to $k \to \infty$).

Representation of the Quon Algebras

We can find several Hilbertian representations of the algebras A_1 and A_2. In this work, we take the representation of $A_1 \otimes A_2$ defined by the following action

$$a_{1+}|n_1,n_2) = |n_1 + 1, n_2), \quad a_{1+}|k - 1, n_2) = 0 \tag{9}$$

$$a_{1-}|n_1,n_2) = [n_1]_q |n_1 - 1, n_2), \quad a_{1-}|0, n_2) = 0 \tag{10}$$

$$a_{2+}|n_1,n_2) = [n_2 + 1]_q |n_1, n_2 + 1), \quad a_{2+}|n_1, k - 1) = 0 \tag{11}$$

$$a_{2-}|n_1,n_2) = |n_1, n_2 - 1), \quad a_{2-}|n_1, 0) = 0 \tag{12}$$

$$N_1|n_1,n_2) = n_1|n_1,n_2), \quad N_2|n_1,n_2) = n_2|n_1,n_2) \tag{13}$$

on a finite (Fock) space $\mathcal{F}_k = \{|n_1,n_2) : n_1, n_2 = 0, 1, ..., k - 1\}$ of dimension dim $\mathcal{F}_k = k^2$. In Eqs. (*10*) and (*11*), we use

$$[x]_q = \frac{1 - q^x}{1 - q} \quad \text{for} \quad x \in \mathbf{R} \tag{14}$$

which yields

$$[n]_q = 1 + q + ... + q^{n-1} \quad \text{for} \quad n \in N^* \quad \text{and} \quad [0]_q = 0 \tag{15}$$

as a particular case. We shall also use the *q*-deformed factorial defined by

$$[n]_q! = [1]_q[2]_q \dots [n]_q \quad \text{for} \quad n \in \mathbf{N}^*, \quad [0]_q! = 1 \tag{16}$$

so that $[n+1]_q! = [n]_q![n+1]_q$ for $n \in \mathbf{N}$.

The space $\mathcal{F}_k$ is a unitary space with a scalar product noted $(\,|\,)$. The k^2 vectors $|n_1, n_2)$ are taken in a form such that

$$(n_1', n_2' | n_1, n_2) = \delta(n_1', n_1)\, \delta(n_2', n_2) \tag{17}$$

(i.e., they constitute an orthonormalized basis of $\mathcal{F}_k$). The space $\mathcal{F}_k$ turns out to be the direct product $\mathcal{F}(1) \otimes \mathcal{F}(2)$ of two truncated Fock spaces $\mathcal{F}(i) = \{|n_i) : n_i = 0, 1, \dots, k-1\}$ of dimension dim $\mathcal{F}(i) = k$ $(i = 1, 2)$ corresponding to two truncated harmonic oscillators. At this stage, we realize why the cases $k = 0$ and $k = 1$ should be excluded. The case $k = 1$ would give trivial algebras A_i with $a_{i-} = a_{i+} = 0$ $(i = 1, 2)$ and the case $k = 0$ would lead to a nondefined value of q.

Two Important Operators

We now define the two linear operators

$$H = \sqrt{N_1 (N_2 + 1)} \tag{18}$$

and

$$U_r = \left[a_{1+} + \mathrm{e}^{\frac{1}{2} \mathrm{i} \phi_r} \frac{1}{[k-1]_q!} (a_{1-})^{k-1} \right] \left[a_{2-} + \mathrm{e}^{\frac{1}{2} \mathrm{i} \phi_r} \frac{1}{[k-1]_q!} (a_{2+})^{k-1} \right] \tag{19}$$

where the arbitrary real parameter ϕ_r is taken in the form

$$\phi_r = \pi(k-1)r \quad \text{with} \quad r \in \mathbf{R}. \tag{20}$$

It is immediate to show that the action of H and U_r on $\mathcal{F}_k$ is given by

$$H|n_1, n_2) = \sqrt{n_1 (n_2 + 1)}\, |n_1, n_2)$$

$$\text{for} \quad n_i = 0, 1, 2, \dots, k-1 \quad \text{with} \quad i = 1, 2 \tag{21}$$

and

$$U_r|n_1, n_2) = |n_1 + 1, n_2 - 1) \quad \text{for} \quad n_1 \neq k-1 \quad \text{and} \quad n_2 \neq 0 \tag{22}$$

$$U_r|k-1,n_2) = \mathrm{e}^{\frac{1}{2}\mathrm{i}\phi_r}|0,n_2-1) \quad \text{for} \quad n_2 \neq 0 \tag{23}$$

$$U_r|n_1,0) = \mathrm{e}^{\frac{1}{2}\mathrm{i}\phi_r}|n_1+1,k-1) \quad \text{for} \quad n_1 \neq k-1 \tag{24}$$

$$U_r|k-1,0) = \mathrm{e}^{\mathrm{i}\phi_r}|0,k-1) \quad \text{for} \quad n_1 = k-1 \quad \text{and} \quad n_2 = 0 . \tag{25}$$

The operators H and U_r satisfy interesting properties. First, it is obvious that the operator H is Hermitean. Second, the operator U_r is unitary. In addition, the action of U_r on the space $\mathcal{F}_k$ is cyclic. More precisely, we can check that

$$(U_r)^k = \mathrm{e}^{\mathrm{i}\phi_r} I \tag{26}$$

where I is the identity operator.

From the Schwinger work on angular momentum[12], we introduce

$$J = \frac{1}{2}(n_1+n_2), \quad M = \frac{1}{2}(n_1-n_2) . \tag{27}$$

Consequently, we can write

$$|n_1,n_2) = |J+M, J-M) . \tag{28}$$

We shall use the notation

$$|JM\rangle \equiv |J+M, J-M) \tag{29}$$

for the vector $|J+M, J-M)$. For a fixed value of J, the label M can take $2J+1$ values $M = -J, -J+1, \ldots, J$. Equations (28) and (29) proved to be of central importance for the connection between angular momentum and a coupled pair of ordinary harmonic oscillators[12]. We guess here that they shall play an important role for connecting the Lie algebra of su(2) to a coupled pair of truncated harmonic oscillators.

For fixed k, the maximum value of J is

$$J = J_{\max} = k-1 \tag{30}$$

and the following value of J

$$J = j = \frac{1}{2}(k-1) \tag{31}$$

is admissible. For a given value of $k \in \mathbf{N} \backslash \{0,1\}$, the $2j + 1 = k$ vectors $|jm\rangle$ belong to the vector space $\mathcal{F}_k$. Let $\varepsilon(j)$ be the subspace of $\mathcal{F}_k$, of dimension $\dim \varepsilon(j) = k$, spanned by the k vectors $|jm\rangle$. We can thus associate the space

$$\varepsilon(j) = \{|jm\rangle : m = -1, -j+1, ..., j\} \tag{32}$$

for $j = \frac{1}{2}, 1, \frac{3}{2}, ...$ to the values $k = 2, 3, 4, ...,$ respectively. The case $\varepsilon(j = 0)$ can be seen to correspond to the limiting situation where $k \to \infty$.

The action of the operators H and U_r on the space $\varepsilon(j)$ can be described by

$$H|jm\rangle = \sqrt{(j+m)(j-m+1)}\,|jm\rangle \tag{33}$$

and

$$U_r|jm\rangle = [1 - \delta(m,j)]\,|jm+1\rangle + \delta(m,j)e^{i\phi_r}\,|j-j\rangle \tag{34}$$

which are a simple rewriting, in terms of the vectors $|jm\rangle$, of Eqs (21) and (22)–(25), respectively. The subspace $\varepsilon(j)$ of $\mathcal{F}_k$ is thus stable under H and U_r. Furthermore, the action of the adjoint $U_r^{\dagger}$ of U_r on the space $\varepsilon(j)$ is given by

$$U_r^{\dagger}|jm\rangle = [1 - \delta(m,-j)]\,|jm-1\rangle + \delta(m,-j)e^{-i\phi_r}\,|jj\rangle . \tag{35}$$

We can check that the operator H is Hermitean and the operator U_r is unitary on the space $\varepsilon(j)$. Equation (26) can be rewritten as

$$(U_r)^{2j+1} = e^{i\phi_r}\,I \tag{36}$$

which reflects the cyclic character of U_r on $\varepsilon(j)$.

Finally let us mention that, as far as the operators H, U_r and $U_r^{\dagger}$ act on the space $\varepsilon(j)$, one can write

$$H = \sum_{m=-j}^{j} \sqrt{(j+m)(j-m+1)}\,|jm\rangle\langle jm| \tag{37}$$

$$U_r = \sum_{m=-j}^{j-1} |jm+1\rangle\langle jm| + e^{+i\phi_r}\,|j-j\rangle\langle jj| \tag{38}$$

$$U_r^{\dagger} = \sum_{m=-j+1}^{j} |jm-1\rangle\langle jm| + e^{-i\phi_r}\,|jj\rangle\langle j-j| \tag{39}$$

where we have introduced a la Dirac projectors on $\varepsilon(j)$.

The SU(2) Generators

We are now in a position to give a realization of the Lie algebra of the group SU(2) in terms of the generators of A_1 and A_2. Let us define the three operators

$$J_+ = H U_r \,, \quad J_- = U_r^\dagger H \tag{40}$$

and

$$J_z = \frac{1}{2}(N_1 - N_2) \,. \tag{41}$$

It is straightforward to check that the action on the vector $|jm\rangle$ of the operators defined by Eqs. (*40*) and (*41*) is given by

$$J_+ |jm\rangle = \sqrt{(j-m)(j+m+1)}\,|jm+1\rangle \tag{42}$$

$$J_- |jm\rangle = \sqrt{(j+m)(j-m+1)}\,|jm-1\rangle \tag{43}$$

and

$$J_z |jm\rangle = m|jm\rangle \,. \tag{44}$$

Consequently, we have the commutation relations

$$[J_z , J_+] = +J_+ \,, \quad [J_z , J_-] = -J_- \,, \quad [J_+ , J_-] = 2J_z \tag{45}$$

which correspond to the Lie algebra of SU(2).

We have here an unusual result for Lie algebras. In the context of deformations, we generally start from a Lie algebra, then deform it and finally find a realization in terms of deformed oscillator algebras. Here we started from two q-deformed oscillator algebras from which we derived the nondeformed Lie algebra su(2).

3. AN ALTERNATIVE BASIS FOR THE REPRESENTATION OF SU(2)

An Alternative to the $\{J^2, J_z\}$ Scheme

The decomposition (*40*) of the shift operators J_+ and J_- in terms of H and U_r coincides with the polar decomposition introduced in lit.[13] in a completely different way. This is easily seen by taking the matrix elements of U_r and H

in the $\{J^2, J_z\}$ quantization scheme and by comparing these elements to the ones of the operators Υ and J_T in lit.[13]. We are thus left with

$$H = J_T \tag{46}$$

and, by identifying the arbitrary phase φ of lit.[13] with $\phi_r = 2\pi jr = \pi(k-1)r$, we obtain that

$$U_r = \Upsilon \tag{47}$$

so that Eq. (40) corresponds to $J_+ = J_T \Upsilon$ and $J_- = \Upsilon^\dagger J_T$.

It is immediate to check that the Casimir operator

$$J^2 = \frac{1}{2}(J_+ J_- + J_- J_+) + J_z^2 \tag{48}$$

of su(2) can be rewritten as

$$J^2 = H^2 + J_z^2 - J_z = U_r^\dagger H^2 U_r + J_z^2 + J_z \tag{49}$$

or

$$J^2 = \frac{1}{4}(N_1 + N_2)(N_1 + N_2 + 2) \tag{50}$$

in terms of the generators N_1 and N_2 of A_1 and A_2, respectively. It is a simple matter of calculation to prove that J^2 commutes with U_r for any value of r. (Note that the commutator $[U_r, U_s]$ is different from zero for $r \neq s$.) Therefore, for r fixed, the commuting set $\{J^2, U_r\}$ provides us with an alternative to the familiar commuting set $\{J^2, J_z\}$ of angular momentum theory. It is to be observed that the operators J^2 and U_r can be expressed as functions of the generators of A_1 and A_2 (see Eqs. (19) and (50)).

Eigenvalues and Eigenvectors

The next step is to determine the eigenvalues and eigenvectors of U_r. The eigenvalues and the common eigenvectors of the complete set of commuting operators $\{J^2, U_r\}$ can be easily found. This leads to the following result. The spectra of the operators U_r and J^2 are given by

$$U_r |j\alpha; r\rangle = q^{-\alpha} |j\alpha; r\rangle$$

$$J^2 |j\alpha; r\rangle = j(j+1) |j\alpha; r\rangle \tag{51}$$

where

$$|j\alpha;r\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} q^{\alpha m} |jm\rangle \qquad (52)$$

with the range of values

$$\alpha = -jr, -jr+1, \ldots, -jr+2j, \quad 2j \in \mathbf{N}, \quad r \in \mathbf{R} \qquad (53)$$

modulo $2j + 1$. The parameter q in Eqs. (51) and (52) is

$$q = \exp\left(i\frac{2\pi}{2j+1}\right) \qquad (54)$$

(cf. Eq. (8) with $k = 2j + 1$ for $k \in \mathbf{N}\setminus\{0,1\}$ and $k \to \infty$ for $j = 0$).

The label μ used in Section 1 is here of the form $\mu \equiv \alpha;r$ with a fixed value of r. It is important to note that the label α in Eqs. (51) and (52) goes, by step of 1, from $-jr$ to $-jr + 2j$; it is only for $r = 1$ that α goes, by step of 1, from $-j$ to j.

The inter-basis expansion coefficients

$$\langle jm|j\alpha;r\rangle = \frac{1}{\sqrt{2j+1}} q^{\alpha m} = \frac{1}{\sqrt{2j+1}} \exp\left(i\frac{2\pi}{2j+1}\alpha m\right) \qquad (55)$$

(with $m = -j, -j+1, \ldots, j$ and $\alpha = -jr, -jr+1, \ldots, -jr+2j$) in Eq. (52) define a unitary transformation that allows to pass from the well-known orthonormal standard spherical basis

$$S = \{|jm\rangle : 2j \in \mathbf{N}, m = -j, -j+1, \ldots, j\} \qquad (56)$$

to the orthonormal nonstandard basis

$$B_r = \{|j\alpha;r\rangle : 2j \in \mathbf{N}, \alpha = -jr, -jr+1, \ldots, -jr+2j\} \qquad (57)$$

for the space

$$\varepsilon = \bigoplus_{j=0,\frac{1}{2},1,\ldots} \varepsilon(j) \qquad (58)$$

where $\varepsilon(j)$ is a subspace of constant angular momentum j (see Eq. (32)). For fixed r, the expansion coefficients satisfy the unitarity property

$$\sum_{m=-j}^{j} \langle jm|j\alpha;r\rangle^* \langle jm|j\alpha';r\rangle = \delta(\alpha',\alpha) \tag{59}$$

and

$$\sum_{\alpha=-jr}^{-j+2j} \langle jm|j\alpha;r\rangle \langle jm'|j\alpha;r\rangle^* = \delta(m',m) \ . \tag{60}$$

Then, the development

$$|jm\rangle = \frac{1}{\sqrt{2j+1}} \sum_{\alpha=-jr}^{-jr+2j} q^{-m\alpha} |j\alpha;r\rangle \tag{61}$$

with

$$m = -j, -j+1, ..., j, \quad 2j \in \mathbf{N} \tag{62}$$

is the inverse of Eq. (*52*) and makes it possible to pass from the nonstandard basis B_r to the standard basis S.

The representation theory of SU(2) can be transcribed in the $\{J^2, U_r\}$ scheme. In this scheme, the rotation matrix elements for the rotation R of SO(3) assumes the form

$$\mathcal{D}_r^{(j)}(R)_{\alpha\alpha'} = \frac{1}{2j+1} \sum_{m=-j}^{j} \sum_{m'=-j}^{j} q^{-\alpha m + \alpha' m'} \mathcal{D}^{(j)}(R)_{mm'} \tag{63}$$

in terms of the standard matrix elements $\mathcal{D}^{(j)}(R)_{mm'}$. Then, the behavior of the vector $|j\alpha;r\rangle$ under an arbitrary rotation R is given by

$$P_R |j\alpha;r\rangle = \sum_{\alpha'} |j\alpha';r\rangle \mathcal{D}_r^{(j)}(R)_{\alpha'\alpha} \tag{64}$$

where P_R stands for the operator associated with R. If R is a rotation around the z-axis, Eq. (*64*) takes a simple form. Indeed, if $R(\varphi)$ is a rotation of an angle

$$\varphi = p\frac{2\pi}{2j+1} \quad \text{with} \quad p = 0, 1, 2, ..., 2j \tag{65}$$

around the z-axis, we have

$$P_{R(\varphi)} |j\alpha;r\rangle = |j\alpha';r\rangle \tag{66}$$

where

$$\alpha' = \alpha - p, \quad \mathrm{mod}(2j+1) \ . \tag{67}$$

Consequently, the set $\{|j\alpha;r\rangle : \alpha = -jr, -jr + 1, \ldots, -jr + 2j\}$ spans a representation of dimension $2j + 1$ of the cyclic subgroup C_{2j+1} of SO(3). It can be seen that this representation is nothing but the regular representation of C_{2j+1}. The nonstandard basis B_r presents some characteristics of a group–subgroup type basis in the sense that the set $\{|j\alpha;r\rangle : \alpha = -jr, -jr + 1, \ldots, -jr + 2j\}$ carries a representation of a subgroup of SO(3). However, this representation is reducible except for $j = 0$. Therefore, the label $\mu \equiv \alpha;r$ does not correspond to some irreducible representation of a subgroup of SU(2) or SO(3) $\equiv$ SU(2)/Z_2 so that the basis B_r also exhibits some characteristics of a nongroup–subgroup type basis.

The behavior of the vector $|j\alpha;r\rangle$ under the time-reversal operator K is given by

$$K|j\alpha;r\rangle = \sum_{\alpha'} \begin{pmatrix} j & j \\ \alpha & \alpha' \end{pmatrix}_r |j\alpha';r\rangle \tag{68}$$

where

$$\begin{pmatrix} j & j \\ \alpha & \alpha' \end{pmatrix}_r = \frac{1}{2j+1} \sum_{m=-j}^{j} \sum_{m'=-j}^{j} q^{-\alpha m + \alpha' m'} \begin{pmatrix} j & j \\ m & m' \end{pmatrix}. \tag{69}$$

Here, the 2-jm symbol (also called a 1-jm symbol for evident reasons) reads

$$\begin{pmatrix} j & j \\ m & m' \end{pmatrix} = (-1)^{j+m} \delta(m',-m) \tag{70}$$

and defines the metric tensor introduced by Wigner[6]. (The normalization chosen for the Wigner metric tensor is the one of Edmonds[14].)

The 2-$j\alpha$ metric tensor allows us to pass from a given irreducible representation matrix of SU(2) to its complex conjugate. Indeed, we have

$$\mathcal{D}_r^{(j)}(R)^*_{\beta\beta'} = \sum_{\alpha\alpha'} \begin{pmatrix} j & j \\ \beta & \alpha \end{pmatrix}_r^* \mathcal{D}_r^{(j)}(R)_{\alpha\alpha'} \begin{pmatrix} j & j \\ \beta' & \alpha' \end{pmatrix}_r \tag{71}$$

(the two j's in the 2-$j\alpha$ metric tensor are identical because the irreducible representation class (j) of SU(2) is identical to its complex conjugate).

For any value of r, the basis B_r is an alternative to the spherical basis S of the space ε. Two bases B_r and B_s with $r \neq s$ are thus two equally admissible orthonormal bases for ε. The vectors of the bases B_r and B_s are common eigenvectors of $\{J^2, U_r\}$ and $\{J^2, U_s\}$, respectively. The overlap between the bases B_r and B_s is controlled by

$$\langle j'\alpha;r|j\beta;s\rangle = \delta(j',j) \frac{1}{2j+1} \frac{\sin(\alpha-\beta)\pi}{\sin(\alpha-\beta)\dfrac{\pi}{2j+1}} \tag{72}$$

with $\alpha = -jr, -jr + 1, \ldots, -jr + 2j$ and $\beta = -js, -js + 1, \ldots, -js + 2j$.

Some Examples

As an illustration, we continue with some examples concerning the subspaces $\varepsilon(\frac{1}{2})$ and $\varepsilon(1)$.

The case $j = \frac{1}{2}$. For $r = 1$, Eq. (*52*) gives

$$\left|\frac{1}{2}-\frac{1}{2};1\right\rangle = \frac{1}{\sqrt{2}}\left(\rho\left|\frac{1}{2}-\frac{1}{2}\right\rangle + \rho^{-1}\left|\frac{1}{2}+\frac{1}{2}\right\rangle\right)$$

$$\left|\frac{1}{2}+\frac{1}{2};1\right\rangle = \frac{1}{\sqrt{2}}\left(\rho^{-1}\left|\frac{1}{2}-\frac{1}{2}\right\rangle + \rho\left|\frac{1}{2}+\frac{1}{2}\right\rangle\right). \tag{73}$$

where $\rho = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$. For $r = 0$, we have

$$\left|\frac{1}{2}0;0\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}-\frac{1}{2}\right\rangle + \left|\frac{1}{2}+\frac{1}{2}\right\rangle\right)$$

$$\left|\frac{1}{2}1;0\right\rangle = \frac{1}{\sqrt{2}}\left(\rho^{-2}\left|\frac{1}{2}-\frac{1}{2}\right\rangle + \rho^{2}\left|\frac{1}{2}+\frac{1}{2}\right\rangle\right) \tag{74}$$

The case $j = 1$. By putting $\omega = \mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ we obtain

$$|1-1;1\rangle = \frac{1}{\sqrt{3}}\left(\omega|1-1\rangle + |10\rangle + \omega^{-1}|1+1\rangle\right)$$

$$|10;1\rangle = \frac{1}{\sqrt{3}}\left(|1-1\rangle + |10\rangle + |1+1\rangle\right)$$

$$|1+1;1\rangle = \frac{1}{\sqrt{3}}\left(\omega^{-1}|1-1\rangle + |10\rangle + \omega|1+1\rangle\right) \tag{75}$$

for $r = 1$ and

$$|10;0\rangle = \frac{1}{\sqrt{3}}\left(|1-1\rangle + |10\rangle + |1+1\rangle\right)$$

$$|11;0\rangle = \frac{1}{\sqrt{3}}\left(\omega^{-1}|1-1\rangle + |10\rangle + \omega|1+1\rangle\right)$$

$$|12;0\rangle = \frac{1}{\sqrt{3}}\left(\omega|1-1\rangle + |10\rangle + \omega^{-1}|1+1\rangle\right) \tag{76}$$

for $r = 0$.

We thus foresee that it is quite possible to achieve the construction of the WRa of the group SU(2) in the $\{J^2, U_r\}$ scheme. This furnishes an alternative to the WRa of SU(2) in the SU(2) $\supset$ U(1) basis corresponding to the $\{J^2, J_z\}$ scheme.

4. A NEW APPROACH TO THE WIGNER–RACAH ALGEBRA OF SU(2)

In this section, we give the basic ingredients for the WRa of SU(2) in the $\{J^2, U_r\}$ scheme. The Clebsch–Gordan coefficients (CGc's) or coupling coefficients adapted to the $\{J^2, U_r\}$ scheme are defined from the SU(2) $\supset$ U(1) CGc's adapted to the $\{J^2, J_z\}$ scheme. The adaptation to the $\{J^2, U_r\}$ scheme afforded by Eq. (52) is transferred to SU(2) irreducible tensor operators. This yields the Wigner–Eckart theorem in the $\{J^2, U_r\}$ scheme.

Coupling Coefficients in the $\{J^2, U_r\}$ Scheme

When passing from the $\{J^2, J_z\}$ scheme to the $\{J^2, U_r\}$ scheme, the CGc's $(j_1 j_2 m_1 m_2 | j_3 m_3)$ are replaced by the coefficients

$$(j_1 j_2 \alpha_1 \alpha_2 | j_3 \alpha_3)_r = \frac{1}{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}$$

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} q_1^{-\alpha_1 m_1} q_2^{-\alpha_2 m_2} q_3^{-\alpha_3 m_3} (j_1 j_2 m_1 m_2 | j_3 m_3) \tag{77}$$

where the q_a's are given in terms of j_a by

$$q_a = \exp\left(i\frac{2\pi}{2j+1}\right), \quad a = 1, 2, 3 \tag{78}$$

(cf. Eq. (54)).

The new CGc's $(j_1 j_2 \alpha_1 \alpha_2 | j\alpha)_r$ in the $\{J^2, U_r\}$ scheme are simple linear combinations of the SU(2) $\supset$ U(1) CGc's. The symmetry properties of the coupling coefficients $(j_1 j_2 \alpha_1 \alpha_2 | j\alpha)_r$ cannot be expressed in a simple way (except the symmetry under the interchange $j_1 \alpha_1 \leftrightarrow j_2 \alpha_2$). Let us introduce the f_r symbol via

$$
f_r \begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} = (-1)^{2 j_3} \frac{1}{\sqrt{2j_1 + 1}} (j_2 j_3 \alpha_2 \alpha_3 | j_1 \alpha_1)_r^* \; . \tag{79}
$$

Its value is multiplied by the factor $(-1)^{j_1 + j_2 + j_3}$ when its two last columns are interchanged. However, the interchange of two other columns cannot be described by a simple symmetry property. Nevertheless, the f_r symbol is of central importance for the calculation of matrix elements of irreducible tensor operators via the Wigner–Eckart theorem in the $\{J^2, U_r\}$ scheme (see Eq. (*106*) below).

Following lit.[9], we define a more symmetrical symbol, namely the $\overline{f_r}$ symbol, through

$$
\overline{f_r} \begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} = \frac{1}{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)}}
$$

$$
\sum_{m_1 = -j_1}^{j_1} \sum_{m_2 = -j_2}^{j_2} \sum_{m_3 = -j_3}^{j_3} q_1^{-\alpha_1 m_1} q_2^{-\alpha_2 m_2} q_3^{-\alpha_3 m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} . \tag{80}
$$

The 3-*jm* symbol on the right-hand side of Eq. (*80*) is an ordinary Wigner symbol for the SU(2) group in the SU(2) $\supset$ U(1) basis. It is possible to pass from the f_r symbol to the $\overline{f_r}$ symbol and vice versa by means of the metric tensor introduced in Section 3. Indeed, we can check that

$$
\overline{f_r} \begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} = \sum_{\alpha_3'} \begin{pmatrix} j_3 & j_3 \\ \alpha_3 & \alpha_3' \end{pmatrix}_r f_r \begin{pmatrix} j_3 & j_2 & j_1 \\ \alpha_3' & \alpha_2 & \alpha_1 \end{pmatrix}^* \tag{81}
$$

or alternatively

$$
f_r \begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} = \sum_{\alpha_1'} \begin{pmatrix} j_1 & j_1 \\ \alpha_1' & \alpha_1 \end{pmatrix}_r \overline{f_r} \begin{pmatrix} j_1 & j_3 & j_2 \\ \alpha_1' & \alpha_3 & \alpha_2 \end{pmatrix}^* . \tag{82}
$$

The $\overline{f_r}$ symbol is more symmetrical than the f_r one. The $\overline{f_r}$ symbol exhibits the same symmetry properties under permutations of its columns as the 3-*jm* Wigner symbol: Its value is multiplied by $(-1)^{j_1 + j_2 + j_3}$ under an odd per-

mutation and does not change under an even permutation. In other words, we have

$$\overline{f_r}\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} = \varepsilon_{abc} \; \overline{f_r}\begin{pmatrix} j_a & j_b & j_c \\ \alpha_a & \alpha_b & \alpha_c \end{pmatrix} \tag{83}$$

where $\varepsilon_{abc} = 1$ or $(-1)^{j_1+j_2+j_3}$ according to whether *abc* corresponds to an even or odd permutation of 123.

The orthogonality properties of the highly symmetrical $\overline{f_r}$ symbol easily follow from the corresponding properties of the 3-*jm* Wigner symbol. Thus, we have

$$\sum_{j_3\alpha_3} (2j_3 + 1) \, \overline{f_r}\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}^* \; \overline{f_r}\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1' & \alpha_2' & \alpha_3 \end{pmatrix} =$$

$$= \delta(\alpha_1', \alpha_1)\, \delta(\alpha_2', \alpha_2) \tag{84}$$

and

$$\sum_{\alpha_1\alpha_2} \overline{f_r}\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \overline{f_r}\begin{pmatrix} j_1 & j_2 & j_3' \\ \alpha_1 & \alpha_2 & \alpha_3' \end{pmatrix}^* =$$

$$= \frac{1}{2j_3 + 1} \Delta(0|j_1 \otimes j_2 \otimes j_3)\, \delta(j_3', j_3)\, \delta(\alpha_3', \alpha_3) \tag{85}$$

where $\Delta(0|j_1 \otimes j_2 \otimes j_3) = 1$ or 0 according to whether the Kronecker product $(j_1) \otimes (j_2) \otimes (j_3)$ contains or does not contain the identity irreducible representation class (0) of SU(2). Note that the real number *r* is the same for all the $\overline{f_r}$ symbols occurring in Eqs (*84*) and (*85*).

The values of the SU(2) CGc's in the $\{J^2, U_r\}$ scheme as well as of the f_r and $\overline{f_r}$ coefficients are not necessarily real numbers. For instance, we have the following property under complex conjugation

$$\overline{f_r}\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1' & \alpha_2' & \alpha_3' \end{pmatrix}^* = \sum_{\alpha_1\alpha_2\alpha_3} \begin{pmatrix} j_1 & j_1 \\ \alpha_1' & \alpha_1 \end{pmatrix}_r^* \begin{pmatrix} j_2 & j_2 \\ \alpha_2' & \alpha_2 \end{pmatrix}_r^* \begin{pmatrix} j_3 & j_3 \\ \alpha_3' & \alpha_3 \end{pmatrix}_r^*$$

$$\overline{f_r}\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}. \tag{86}$$

Then, the behavior of the $\overline{f_r}$ symbol under complex conjugation is completely different from the one of the ordinary 3-jm Wigner symbol. In this respect, we have

$$\overline{f_r}\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}^* = (-1)^{j_1+j_2+j_3}\, \overline{f_r}\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}. \tag{87}$$

Hence, the value of the $\overline{f_r}$ coefficient is real if $j_1 + j_2 + j_3$ is even and pure imaginary if $j_1 + j_2 + j_3$ is odd.

It is to be noted that the 2-$j\alpha$ symbol introduced in Section 3 is a particular case of the $\overline{f_r}$ symbol since we have

$$\begin{pmatrix} j & j \\ \alpha & \alpha' \end{pmatrix}_r = \sqrt{2j+1}\, \overline{f_r}\begin{pmatrix} j & 0 & j \\ \alpha & 0 & \alpha' \end{pmatrix}. \tag{88}$$

Consequently, the orthogonality property

$$\sum_\alpha \begin{pmatrix} j & j \\ \alpha & \beta \end{pmatrix}_r \begin{pmatrix} j & j \\ \alpha & \beta' \end{pmatrix}_r^* = \delta(\beta',\beta) \tag{89}$$

and the symmetry property

$$\begin{pmatrix} j & j \\ \alpha' & \alpha \end{pmatrix}_r = (-1)^{2j}\begin{pmatrix} j & j \\ \alpha & \alpha' \end{pmatrix}_r \tag{90}$$

follow from the corresponding properties of the $\overline{f_r}$ symbol.

The case $r = 1$ deserves a special attention. In that case, we have specific relations because the label α may be 0 for j integer. For example, the value of

$$\overline{f_r}\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}$$

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_3=-j_3}^{j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{91}$$

is equal to 0 if $j_1 + j_2 + j_3$ is odd.

Recoupling Coefficients in the {J^2, U_r} Scheme

The recoupling coefficients of the SU(2) group are rotational invariants[14]. Therefore, they can be expressed in terms of coupling coefficients of SU(2) in the {J^2, U_r} scheme. For example, the 9-j symbol can be expressed in terms of $\overline{f}_r$ symbols by replacing, in its decomposition in terms of 3-jm symbols, the 3-jm symbols by $\overline{f}_r$ symbols. On the other hand, the decomposition of the 6-j symbol in terms of $\overline{f}_r$ symbols requires the introduction of six metric tensors corresponding to the six arguments of the 6-j symbol. These matters shall be developed by following the approach initiated in lit.[9]

We start with the case of the 6-j symbol. Relations involving the 6-j Wigner symbol (or $\overline{W}$ Fano and Racah coefficient[7]) and $\overline{f}_r$ symbols, with four $\overline{f}_r$ symbols, can be easily derived. First, the 6-j symbol can be expressed as

$$
\overline{W}\begin{pmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{pmatrix} = \sum_{\text{all } \alpha'} \sum_{\text{all } \alpha} \begin{pmatrix} j_1 & j_1 \\ \alpha_1 & \alpha'_1 \end{pmatrix}^*_r \begin{pmatrix} j_2 & j_2 \\ \alpha_2 & \alpha'_2 \end{pmatrix}^*_r \begin{pmatrix} j_3 & j_3 \\ \alpha_3 & \alpha'_3 \end{pmatrix}^*_r
$$

$$
\begin{pmatrix} j_4 & j_4 \\ \alpha_4 & \alpha'_4 \end{pmatrix}^*_r \begin{pmatrix} j_5 & j_5 \\ \alpha_5 & \alpha'_5 \end{pmatrix}^*_r \begin{pmatrix} j_6 & j_6 \\ \alpha_6 & \alpha'_6 \end{pmatrix}^*_r
$$

$$
\overline{f}_r\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \overline{f}_r\begin{pmatrix} j_1 & j_5 & j_6 \\ \alpha'_1 & \alpha_5 & \alpha'_6 \end{pmatrix}
$$

$$
\overline{f}_r\begin{pmatrix} j_4 & j_2 & j_6 \\ \alpha'_4 & \alpha'_2 & \alpha_6 \end{pmatrix} \overline{f}_r\begin{pmatrix} j_4 & j_5 & j_3 \\ \alpha_4 & \alpha'_5 & \alpha'_3 \end{pmatrix} \tag{92}
$$

which involves 0+4 $\overline{f}_r$ symbols (no $\overline{f}_r$ symbol on the left-hand side and four on the right-hand side). With the help of Eq. (*86*), Eq. (*92*) can be rewritten as

$$
\overline{W}\begin{pmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{pmatrix} = \sum_{\alpha'_4 \alpha'_5 \alpha'_6} \sum_{\text{all } \alpha} \begin{pmatrix} j_4 & j_4 \\ \alpha_4 & \alpha'_4 \end{pmatrix}^*_r \begin{pmatrix} j_5 & j_5 \\ \alpha_5 & \alpha'_5 \end{pmatrix}^*_r \begin{pmatrix} j_6 & j_6 \\ \alpha_6 & \alpha'_6 \end{pmatrix}^*_r
$$

$$
\overline{f}_r\begin{pmatrix} j_1 & j_3 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}^* \overline{f}_r\begin{pmatrix} j_1 & j_5 & j_6 \\ \alpha_1 & \alpha_5 & \alpha'_6 \end{pmatrix}
$$

$$
\overline{f}_r\begin{pmatrix} j_4 & j_2 & j_6 \\ \alpha'_4 & \alpha_2 & \alpha_6 \end{pmatrix} \overline{f}_r\begin{pmatrix} j_4 & j_5 & j_3 \\ \alpha_4 & \alpha'_5 & \alpha_3 \end{pmatrix}. \tag{93}
$$

An expression involving 1+3 $\overline{f_r}$ symbols is

$$\overline{f_r}\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \overline{W}\begin{pmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{pmatrix} = \Delta(0|j_1 \otimes j_2 \otimes j_3)$$

$$\sum_{\alpha_4' \alpha_5' \alpha_6'} \sum_{\alpha_4 \alpha_5 \alpha_6} \begin{pmatrix} j_4 & j_4 \\ \alpha_4 & \alpha_4' \end{pmatrix}_r^* \begin{pmatrix} j_5 & j_5 \\ \alpha_5 & \alpha_5' \end{pmatrix}_r^* \begin{pmatrix} j_6 & j_6 \\ \alpha_6 & \alpha_6' \end{pmatrix}_r^*$$

$$\overline{f_r}\begin{pmatrix} j_1 & j_5 & j_6 \\ \alpha_1 & \alpha_5 & \alpha_6' \end{pmatrix} \overline{f_r}\begin{pmatrix} j_4 & j_2 & j_6 \\ \alpha_4' & \alpha_2 & \alpha_6 \end{pmatrix} \overline{f_r}\begin{pmatrix} j_4 & j_5 & j_3 \\ \alpha_4 & \alpha_5' & \alpha_3 \end{pmatrix}. \tag{94}$$

We also have a 2+2 relationship

$$\sum_{j_3 \alpha_3} (2j_3 + 1)\ \overline{f_r}\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \overline{f_r}\begin{pmatrix} j_4 & j_5 & j_3 \\ \alpha_4 & \alpha_5 & \alpha_3 \end{pmatrix}^* \overline{W}\begin{pmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{pmatrix} =$$

$$= \sum_{\alpha_4' \alpha_5' \alpha_6'} \sum_{\alpha_6} \begin{pmatrix} j_4 & j_4 \\ \alpha_4 & \alpha_4' \end{pmatrix}_r^* \begin{pmatrix} j_5 & j_5 \\ \alpha_5' & \alpha_5 \end{pmatrix}_r^* \begin{pmatrix} j_6 & j_6 \\ \alpha_6 & \alpha_6' \end{pmatrix}_r^*$$

$$\overline{f_r}\begin{pmatrix} j_1 & j_5 & j_6 \\ \alpha_1 & \alpha_5' & \alpha_6' \end{pmatrix} \overline{f_r}\begin{pmatrix} j_4 & j_2 & j_6 \\ \alpha_4' & \alpha_2 & \alpha_6 \end{pmatrix} \tag{95}$$

and a 3+1 relationship

$$\sum_{j_3 \alpha_3} \sum_{\alpha_4' \alpha_5' \alpha_6'} \sum_{\alpha_1 \alpha_5} (2j_3 + 1) \begin{pmatrix} j_4 & j_4 \\ \alpha_4' & \alpha_4 \end{pmatrix}_r \begin{pmatrix} j_5 & j_5 \\ \alpha_5 & \alpha_5' \end{pmatrix}_r \begin{pmatrix} j_6 & j_6 \\ \alpha_6 & \alpha_6' \end{pmatrix}_r$$

$$\overline{f_r}\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \overline{f_r}\begin{pmatrix} j_1 & j_5 & j_6 \\ \alpha_1 & \alpha_5 & \alpha_6' \end{pmatrix}^* \overline{f_r}\begin{pmatrix} j_4 & j_5 & j_3 \\ \alpha_4' & \alpha_5' & \alpha_3 \end{pmatrix}^*$$

$$\overline{W}\begin{pmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{pmatrix} = \frac{1}{2j_6 + 1} \Delta(0|j_1 \otimes j_5 \otimes j_6)\ \overline{f_r}\begin{pmatrix} j_4 & j_2 & j_6 \\ \alpha_4 & \alpha_2 & \alpha_6 \end{pmatrix}. \tag{96}$$

By using the orthonormality of the $\overline{f_r}$ symbol in conjunction with Eq. (*96*), we would obtain a 4+0 relationship which turns out to be the well-known orthonormality relation[7] for the $\overline{W}$ coefficient.

We continue with the 9-j Wigner symbol (or X Fano and Racah coefficient[7]). Relations involving six $\overline{f_r}$ symbols and one 9-j symbol can be obtained in a straightforward way. First, we have the very symmetrical expression of the type 0+6

$$
X\begin{pmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{32} \end{pmatrix} = \sum_{\text{all } \alpha} \overline{f_r}\begin{pmatrix} j_{11} & j_{21} & j_{31} \\ \alpha_{11} & \alpha_{21} & \alpha_{31} \end{pmatrix} \overline{f_r}\begin{pmatrix} j_{12} & j_{22} & j_{32} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \end{pmatrix}
$$

$$
\overline{f_r}\begin{pmatrix} j_{13} & j_{23} & j_{33} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix} \overline{f_r}\begin{pmatrix} j_{11} & j_{12} & j_{13} \\ \alpha_{11} & \alpha_{12} & \alpha_{13} \end{pmatrix}^*
$$

$$
\overline{f_r}\begin{pmatrix} j_{21} & j_{22} & j_{23} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \end{pmatrix}^* \overline{f_r}\begin{pmatrix} j_{31} & j_{32} & j_{33} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}^*. \tag{97}
$$

Other relations with six $\overline{f_r}$ symbols can be derived by combining Eq. (*97*) and the orthonormality relations of the $\overline{f_r}$ symbols. For instance, we have the relation of the type 1+5

$$
\overline{f_r}\begin{pmatrix} j_{31} & j_{32} & j_{33} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} X\begin{pmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{32} \end{pmatrix} = \Delta(0|j_{31} \otimes j_{32} \otimes j_{33})
$$

$$
\sum_{\alpha_{11}\alpha_{12}\alpha_{13}} \sum_{\alpha_{21}\alpha_{22}\alpha_{23}} \overline{f_r}\begin{pmatrix} j_{11} & j_{21} & j_{31} \\ \alpha_{11} & \alpha_{21} & \alpha_{31} \end{pmatrix}
$$

$$
\overline{f_r}\begin{pmatrix} j_{12} & j_{22} & j_{32} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \end{pmatrix} \overline{f_r}\begin{pmatrix} j_{13} & j_{23} & j_{33} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix}
$$

$$
\overline{f_r}\begin{pmatrix} j_{11} & j_{12} & j_{13} \\ \alpha_{11} & \alpha_{12} & \alpha_{13} \end{pmatrix}^* \overline{f_r}\begin{pmatrix} j_{21} & j_{22} & j_{23} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \end{pmatrix}^* \tag{98}
$$

and the relation of the type 2+4

$$
\sum_{j_{31}\alpha_{31}} (2j_{31} + 1)\, \overline{f_r}\begin{pmatrix} j_{11} & j_{21} & j_{31} \\ \alpha_{11} & \alpha_{21} & \alpha_{31} \end{pmatrix}^* \overline{f_r}\begin{pmatrix} j_{31} & j_{32} & j_{33} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} X\begin{pmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{pmatrix} =
$$

$$\sum_{\alpha_{12}\alpha_{13}} \sum_{\alpha_{22}\alpha_{23}} \overline{f}_r\begin{pmatrix} j_{12} & j_{22} & j_{32} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \end{pmatrix} \overline{f}_r\begin{pmatrix} j_{13} & j_{23} & j_{33} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix}$$

$$\overline{f}_r\begin{pmatrix} j_{11} & j_{12} & j_{13} \\ \alpha_{11} & \alpha_{12} & \alpha_{13} \end{pmatrix}^* \overline{f}_r\begin{pmatrix} j_{21} & j_{22} & j_{23} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \end{pmatrix}^* . \tag{99}$$

Relations involving coupling and recoupling coefficients are of considerable interest for the calculation of matrix elements. In particular, $\overline{W}$ and X coefficients occur in matrix elements of scalar product and tensor product of two irreducible tensor operators.

Wigner–Eckart Theorem in the $\{J^2, U_r\}$ Scheme

Irreducible tensor operators. From the spherical components $T_m^{(k)}$ (with $m = -k, -k + 1, \ldots, k$) of an SU(2) irreducible tensor operator $\mathbf{T}^{(k)}$, we define the $2k + 1$ components

$$T_{\alpha;r}^{(k)} = \frac{1}{\sqrt{2k+1}} \sum_{m=-k}^{k} q^{\alpha m} T_m^{(k)} \tag{100}$$

with

$$\alpha = -kr, -kr + 1, \ldots, -kr + 2k, \quad 2k \in \mathbf{N} \tag{101}$$

where r is fixed in $\mathbf{R}$. The behavior of $T_{\alpha;r}^{(k)}$ under a rotation R is described by

$$P_R T_{\alpha;r}^{(k)} P_R^{-1} = \sum_{\alpha'} T_{\alpha';r}^{(k)} \mathcal{D}_r^{(j)}(R)_{\alpha'\alpha} . \tag{102}$$

Following Racah[7], given two SU(2) irreducible tensor operators $\mathbf{T}^{(k_1)}$ and $\mathbf{U}^{(k_2)}$, we can define the tensor product $\{\mathbf{T}^{(k_1)}\mathbf{U}^{(k_2)}\}^{(k)}$ of components

$$\{\mathbf{T}^{(k_1)}\mathbf{U}^{(k_2)}\}_{\alpha;r}^{(k)} = \sum_{\alpha_1\alpha_2} (k_1 k_2 \alpha_1 \alpha_2 \,|\, k\alpha)_r T_{\alpha_1;r}^{(k_1)} U_{\alpha_2;r}^{(k_2)} . \tag{103}$$

As a particular case, we get the scalar product

$$(\mathbf{T}^{(k)} \cdot \mathbf{U}^{(k)}) = (-1)^k \sqrt{2k+1} \{\mathbf{T}^{(k)}\mathbf{U}^{(k)}\}_{0;r}^{(0)} . \tag{104}$$

More specifically, we have

$$(\mathbf{T}^{(k)} \cdot \mathbf{U}^{(k)}) = (-1)^k \sum_{\alpha\alpha'} \begin{pmatrix} k & k \\ \alpha & \alpha' \end{pmatrix}_r T_{\alpha;r}^{(k)} U_{\alpha';r}^{(k)} \tag{105}$$

which can be identified with the scalar product introduced by Racah[7].

Matrix elements of tensor operators. In the $\{J^2, U_r\}$ scheme, the Wigner–Eckart theorem reads

$$\langle \tau_1 j_1 \alpha_1 ;r| T_{\alpha;r}^{(k)} | \tau_2 j_2 \alpha_2 ;r\rangle = (\tau_1 j_1 || T^{(k)} || \tau_2 j_2) \; f_r \begin{pmatrix} j_1 & j_2 & k \\ \alpha_1 & \alpha_2 & \alpha \end{pmatrix} \qquad (106)$$

where $(t_1 j_1 || T^{(k)} || \tau_2 j_2)$ denotes an ordinary reduced matrix element. Such a reduced matrix element is clearly basis-independent. The reduced matrix element in Eq. (*106*) is identical with the one introduced by Racah[7]. It is a rotational invariant that can be in general expressed in terms of basic invariants (e.g., reduced matrix element of Wigner unit operator, $\overline{W}$ and X coefficients). Therefore, it does not depend on the labels α_1, α_2 and α. On the contrary, the $\overline{f}_r$ coefficient in Eq. (*106*), defined by Eq. (*79*), depends on the labels α_1, α_2 and α. The information on the geometry is entirely contained in the $\overline{f}_r$ coefficient.

5. CONCLUDING REMARKS

The main results presented in this paper are the following: (i) The non-deformed Lie algebra su_2 may be constructed from two commuting q-deformed oscillator algebras with q being a root of unity; the latter oscillator algebras are associated with (truncated) harmonic oscillators having a finite number of eigenvectors. (ii) This construction leads to the polar decomposition of the generators J_+ and J_- of SU(2) originally introduced by Lévy–Leblond[13]. (iii) The familiar $\{J^2, J_z\}$ quantization scheme with the (usual) standard spherical basis $\{|jm\rangle : 2j \in \mathbf{N}, \; m = -j, -j + 1, \dots, j\}$, corresponding to the canonical chain of groups SU(2) $\supset$ U(1), is thus replaced by the $\{J^2, U_r\}$ quantization scheme with a (new) basis, namely, the nonstandard basis $B_r = \{|j\alpha;r\rangle : 2j \in \mathbf{N}, \; \alpha = -jr, -jr + 1, \dots, -jr + 2j\}$. (iv) The Wigner–Racah algebra of SU(2) may be developed in the $\{J^2, U_r\}$ scheme.

These various results should be useful in problems involving axial symmetry and in the investigation of quantum mechanics on a finite Hilbert space as developed by several authors[15]. To make the latter point clear, let us write S (see Eq. (*56*)) and B_r (see Eq. (*57*)) as

$$S = \bigcup_{j=0}^{\infty} s^j \qquad (107)$$

and

$$B_r = \bigcup_{j=0}^{\infty} b_r^j \qquad (108)$$

where s^j and b_r^j are two bases that span the subspace $\varepsilon(j)$. It is clear that s^j and b_r^j are two mutually unbiased bases (MUB's) in the sense that

$$|\langle jm|j\alpha;r\rangle| = \frac{1}{\sqrt{\dim \varepsilon(j)}} \ . \tag{109}$$

It is known that the MUB's are especially useful in the theory of quantum information. In this respect, a connection between our results and some of the ones in lit.[15] is presently under study.

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